Last Time: Elementary matrices, matrix inverses.

Ended on a computation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \boxed{\frac{1}{ad-bc}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

NB: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad-bc \neq 0$.

Determinants

The determinant of a matrix is a quantity which tells us if the matrix has an inverse...

→ All matrices are square (i.e. $n \times n$) today...

Defn: The determinant of $n \times n$ matrix M is the sum of the products of entries of M determined by each permutation of the columns [scaled by its sign...]

NB: This condition is a bit weird... we use something called "cofactor expansion" to do actual computations...

Ex (Using Cofactor Expansion): $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xleftarrow{\frac{\text{expand}}{\text{along a row}}} \quad +a(\det[d]) - b(\det[c])$$

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

$$= ad - bc$$ ▨

Ex: Compute $\det(M)$ (using Cofactor expansion) for

$$M = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Sol 1 (Expand along row 1):

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} + & - & + \\ & & \\ & & \end{bmatrix}$$

$$\det(M) = +1 \cdot \det\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} - 2 \det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + 1 \cdot \det\begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}.$$

$$= 1 \cdot (2\cdot 2 - 1 \cdot 2) - 2 \cdot (2 \cdot 2 - 1 \cdot 1) + 1 \cdot (2 \cdot 2 - 2 \cdot 1)$$

$$= 1 \cdot 2 - 2 \cdot 3 + 1 \cdot 2 = 4 - 6 = -2.$$

Sol 2 (Expand along row 3):

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\det\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} = +1 \cdot \det\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} - 2 \cdot \det\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + 2 \cdot \det\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$= 1(2 \cdot 1 - 1 \cdot 2) - 2 \cdot (1 \cdot 1 - 2 \cdot 1) + 2(1 \cdot 2 - 2 \cdot 2)$$

$$= 1 \cdot 0 - 2 \cdot (-1) + 2(-2) = -2$$

Sol 3 (Expand along Column 2):

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} + & - & \\ & + & \\ & - & \end{bmatrix}$$

$$\det\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} = -2 \det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + 2 \det\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - 2 \det\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= -2(2 \cdot 2 - 1 \cdot 1) + 2(1 \cdot 2 - 1 \cdot 1) - 2(1 \cdot 1 - 1 \cdot 2)$$

$$= -6 + 2 - 2(-1) = -6 + 2 + 2 = -2$$

Point: Cofactor Expansion can be done along any row or column to compute the determinant...

Caution: Only use one row or column per expansion...

Ex: Compute $\det\begin{bmatrix} 0 & 2 & 3 & 0 \\ -3 & 2 & 2 & -1 \\ -2 & 2 & -1 & 3 \\ -1 & 3 & 0 & 0 \end{bmatrix}$.

expanding along column 4:

Sol:
$$\det \begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{bmatrix} 0 & 2 & 3 & 0 \\ -3 & 2 & 2 & -1 \\ -2 & 2 & -1 & 3 \\ -1 & 3 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} \\ + \\ \\ \\ \end{matrix}$$

$$= -0\det\begin{bmatrix} -3 & 2 & 2 \\ -2 & 2 & -1 \\ -1 & 3 & 0 \end{bmatrix} + (-1)\det\begin{bmatrix} 0 & 2 & 3 \\ -2 & 2 & -1 \\ -1 & 3 & 0 \end{bmatrix}$$

$$- 3\det\begin{bmatrix} 0 & 2 & 3 \\ -3 & 2 & 2 \\ -1 & 3 & 0 \end{bmatrix} + 0\det\begin{bmatrix} 0 & 2 & 3 \\ -3 & 2 & 2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$= 0 + (-1)\det\begin{bmatrix} 0 & 2 & 3 \\ -2 & 2 & -1 \\ -1 & 3 & 0 \end{bmatrix} - 3\det\begin{bmatrix} 0 & 2 & 3 \\ -3 & 2 & 2 \\ -1 & 3 & 0 \end{bmatrix} + 0$$

$$= (-1)\left(0\det\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} - 2\det\begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix} + 3\det\begin{bmatrix} -2 & 2 \\ -1 & 3 \end{bmatrix}\right)$$

$$-3\left(0\det\begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} - 2\det\begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} + 3\det\begin{bmatrix} -3 & 2 \\ -1 & 3 \end{bmatrix}\right)$$

$$= -\left(0 - 2(0-1) + 3(-6+2)\right)$$

$$-3\left(0 - 2(0+2) + 3(-9+2)\right)$$

$$= -(2-12) - 3(-4-21) = 10 + 75 = 85 \quad \blacksquare$$

Q: What does $\det(M)$ tell us about M?

A: $\det(M) = 0$ if and only if M is not invertible.

i.e. $\det(M) \neq 0$ means M is invertible.

$\rightsquigarrow$ There are formulas for M^{-1} involving $\det(M)$...

$$\left(\text{analogous to } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) \ldots$$

$\rightsquigarrow$ "Hard" exercise: Try for $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ ...

Prop: If M is a square matrix with a zero-row (or column), then $\det(M) = 0$.

Pf: Do cofactor expansion along the zero- (row or column). $\blacksquare$

Ex:

$$\begin{array}{r} \\ \\ \\ \text{zero-row} \rightarrow \\ \\ \end{array} \det \begin{bmatrix} 0 & 1 & 1 & 0 & -1 \\ 0 & 5 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & -7 & e & \pi \end{bmatrix} = 0. \quad \blacksquare$$

NB: The determinant is a function (technically, there is one "determinant function" for each positive integer n):

$$\det : M_{n \times n}(\mathbb{C}) \longrightarrow \mathbb{C}$$

(or $\det : M_{n \times n}(\mathbb{R}) \longrightarrow \mathbb{R}$).

We ~~will~~ can NEVER take determinants of non-square matrices!

Q: What are the determinants of the elementary matrices?

↳ Examples for $n=3$:

$$\det(P_{1,3}) = \det\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = 0 - 0 + 1\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= (0-1) = -1$$

$$\det(P_{2,3}) = \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = 1\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 0 + 0$$
$$= (0-1) = -1$$

verify for yourself: $\det(P_{1,2}) = -1$

Fact: $\det(P_{i,j}) = -1$ for all $i \neq j$ and all n.

What about $M_i(k)$? (i.e. multiply row i by k).

$$\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1\det\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} - 0 + 0$$
$$= 1\cdot(k1 - 0) = k.$$

More generally: for a diagonal matrix:

$$\det\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = a\det\begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} - 0 + 0$$
$$= a(b\cdot c - 0\cdot 0) = abc.$$

Fact: $\det(M_i(k)) = k$.

NB: Pretty easy (using induction and cofactor expansion) to prove the determinant of a diagonal matrix is just the product of it's diagonal entries...

↳ Holds more generally for triangular matrices...

What is the determinant of $A_{i,j}(k)$?

Fact: $\det(A_{i,j}(k)) = 1$ for all $i \neq j$, k.

Point: $M_i(k)$, $P_{i,j}$, and $A_{i,j}(k)$ are the matrices describing row reduction, so we'll see next time how to leverage these facts to make easier computations of $\det(M)$...